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OPERATOR SEMIGROUPS ACTING ON A Γ-SEMIGROUPS

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Abstract

The concept of Γ-semigroup is a generalization of semigroups. In this paper, we briefly introduce the action of left (right) operator semigroups on a Γ-semigroup and deduce in particular that there exists an inclusion preserving bijection between the set of all right ideals of S and the set all right ideals of $L \times S$.

Keywords: Γ-semigroup; Operator semigroup.

1 Introduction

The notion of Γ in algebra was first introduced by Nobusawa [8] as a generalization of ring in the form of Γ-ring. Let M and Γ be additive groups such that for all $a, b, c \in M$ and $\gamma, \beta, \alpha \in \Gamma$, we have $a\gamma b \in M$ and $\gamma a\beta \in \Gamma$ for every a, b, γ and β, then M is called a Γ-ring if the following conditions are satisfied:

(i) $(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b,$
$a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b,$
$a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2,$

(ii) $(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c,$

(iii) if $a\gamma b = 0$ for any $a, b \in M$, then $\gamma = 0$.

The structure of Γ-rings as initiated by [8] was studied by Barnes [1], Luh [7], Ravishankar and Shukla [9], Buys and Groenewald [3], Booth [2] and Kyuno [6]. Motivated by this generalization of a ring, Sen [13] defined the concept of Γ-semigroup. Later, Sen and Saha [14] redefined the Γ-semigroup by weakening slightly the defining conditions of Γ-semigroup to ensure it preserves semigroup structure. The development of Γ-semigroups hinges on the fact that subsets of a semigroup naturally inherits associativity but not necessarily closed. As a result of this, various generalizations and analogues of corresponding results in semigroup theory have been obtained based on the modified definition (see [10, 15, 16, 17]).

In an attempt to broaden the theoretical aspect of Γ-semigroup theory, Dutta and Adhikari [4] slightly changed the defining conditions of Γ-semigroup by Sen and Saha [14] and then introduced the notion of left operator semigroup and right operator semigroup of a Γ-semigroup.

In [4], the authors described the relationship between the set of Γ-ideals and operator semigroups. In relation to the concept, Dutta and Chattopadhyay [5] initiated the notions of uniformly strongly prime semigroup, uniformly strongly prime ideal, Rees congruence on Γ-semigroup and uniformly strongly prime Γ-semigroup and studied these via operator semigroups of Γ-semigroup. Sardar *et al.* [12] showed that the left operator and right

operator semigroups of a Γ-semigroup with unities are Morita equivalence monoid and further established that there is a close connection between the Morita equivalence of monoids and Γ-semigroups. Although there is significantly number of published results in literature on operator semigroups of a Γ-semigroup, however the aspect of operator semigroups acting on a Γ-semigroup observed by [11] has not been given much attention. This serves as a motivation to write this paper and we deduce some results.

2 Preliminaries

We recall some definitions and results related to this paper.

Definition 2.1 *Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exist mappings $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longrightarrow a\alpha b \in S$ and $\Gamma \times S \times \Gamma \longrightarrow \Gamma \mid (\alpha, a, \beta) \longrightarrow \alpha a\beta \in \Gamma$ satisfying the identities $a\alpha(b\beta c) = a(\alpha b\beta)c = (a\alpha b)\beta c$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

The modified definition of Γ-semigroup by Sen and Saha [14] may be regarded as one-sided Γ-semigroup.

Definition 2.2 *Let $S = \{a, b, c...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longrightarrow a\alpha b \in S$ satisfying the property $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

A Γ-semigroup S is called commutative if $x\alpha y = y\alpha x$ for every $x, y \in S$ and $\alpha \in \Gamma$.

Let A and B be two subsets of a Γ-semigroup S. We define the set

$$A\Gamma B = \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}.$$

For simplicity we write $a\Gamma B$, $A\Gamma b$ and $A\gamma B$ instead of $\{a\}\Gamma B$, $A\Gamma\{b\}$ and $A\{\gamma\}B$ respectively.

Let S be an arbitrary semigroup and Γ be a non-empty set. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus, a semigroup can be considered as a Γ-semigroup.

In the following, some examples of Γ-semigroups are presented.

Example 2.1 *Let $S = Z$ be the set of all integers and $\Gamma = \{n \mid n \in N\}$. Then S is a Γ-semigroup with the operation defined by $a\alpha b = a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$.*

Example 2.2 *Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} \mid p \text{ is prime}\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b; then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ-semigroup.*

Example 2.3 *Let $S = \{-i, i, 0\}$ and $\Gamma = S$. Then S is a Γ-semigroup with respect to multiplication of complex numbers whereas S does not reduce to semigroup with respect to multiplication of complex numbers.*

The following definitions and theorems can be found in [5] except Definitions 2.3.

Definition 2.3 *Let S be a Γ-semigroup. A nonempty subset A of S is called left (right) Γ-ideal of S if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$). Further, a non-empty A of a Γ-semigroup S is called Γ-ideal if A is both a left and a right Γ-ideal of S.*

Definition 2.4 *Let S be a Γ-semigroup. Let L and R be the left and right operator semigroups of the Γ-semigroup S. If there exist an element $[e, \delta] \in L$ ($[\delta, e] \in R$) such that $e\delta x = x$ ($x\delta e = x$) for all $x \in S$, then S is said to have the left unity $[e, \delta]$ (right unity $[\delta, e]$).*

Definition 2.5 *Let S be a Γ-semigroup. We define a relation ρ on $S \times \Gamma$ as follows:*

$$(x, \alpha)\rho(y, \beta) \Longleftrightarrow x\alpha s = y\beta s, \ \forall \ s \in S.$$

Obviously ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \ \alpha \in \Gamma\}$. Then L is a semigroup with respect to multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. The semigroup L is called the left operator semigroup of S. Similarly, the right operator semigroup R of a Γ-semigroup S is defined as $R = \{[\alpha, x] : \alpha \in \Gamma, \ x \in S\}$, where $[\alpha, x][\beta, y] = [\alpha, x\beta y]$, for all $x, y \in S$ and $\alpha, \beta \in \Gamma$.

Example 2.4 *Using Example 2.3,*

$S \times \Gamma = \{(-i,-i),(0,0),(i,i),(-i,0),(-i,i),(0,-i),(0,i),(i,-i),(i,0)\}$ *and let* $\rho = \{(-i,-i),(0,0),(i,i),(-i,i),(i,-i)\}$ *be a relation on* $S \times \Gamma$.

By routine calculation, it is obvious that ρ *is an equivalence relation. The equivalence class* $[x,\alpha] = \{(y,\beta) \in \rho \mid (y,\beta)\rho(x,\alpha)\}$. *Therefore,*

$$\begin{aligned}
[-i,-i] &= \{(-i,-i),(i,i)\} \\
[0,0] &= \{(0,0)\} \\
[i,i] &= \{(i,i),(-i,-i)\} \\
[-i,i] &= \{(-i,i),(i,-i)\} \\
[i,-i] &= \{(i,-i),(i,-i)\}
\end{aligned}$$

$\Longrightarrow [-i,-i] = [i,i]$ *and* $[-i,i] = [i,-i]$ *and the set form* $L = \{[-i,-i],[i,i],[-i,i],[i,-i]\}$ *is a left operator semigroup of* S. *Similarly, the right operator semigroup* R *of* S *can be obtained.*

Theorem 2.1 *Let* S *be a* Γ*-semigroup. If* $[e,\delta]$ *is left unity of* S, *then* $[e,\delta]$ *is the identity element of* L.

Theorem 2.2 *Let* S *be a* Γ*-semigroup. If* $[\mu,f]$ *is right unity of* S, *then* $[\mu,f]$ *is the identity element of* R.

Theorem 2.3 *Let* S *be a* Γ*-semigroup with left and right unities and* L *be its left operator semigroup.*

(i) *If* Q *is a* Γ*-ideal of* S, *then* $Q^{+'}$ *is a* Γ*-ideal of* L.

(ii) *If* P *is a* Γ*-ideal of* L, *then* P^{+} *is a* Γ*-ideal of* S.

Theorem 2.4 *Let* S *be a* Γ*-semigroup with left and right unities and* R *be its right operator semigroup.*

(i) *If* Q *is a* Γ*-ideal of* S, *then* $Q^{+'}$ *is a* Γ*-ideal of* R.

(ii) *If* P *is a* Γ*-ideal of* R, *then* P^{*} *is a* Γ*-ideal of* S.

Theorem 2.5 *Let* S *be a* Γ*-semigroup with left and right unities and let* L *and* R *be its left operator semigroup and right operator semigroup respectively. Then there is an inclusion preserving bijection between the set of all* Γ*-ideals of* S *and the set of all* Γ*-ideals of* $L(R)$ *via the mapping* $Q \longrightarrow Q^{+'}$ $(Q \longrightarrow Q^{*'})$ *where* Q *is a* Γ*-ideal of* S.

3 Main Results

Throughout S stands for one-sided Γ-semigroup unless otherwise mentioned.

The following proposition and theorem show the commutativity and isomorphism of operator semigroups.

Proposition 3.1 *Let S be a commutative Γ-semigroup and $\alpha \in \Gamma$. Then the left operator semigroup L and the right operator semigroup R of S are commutative.*

Proof
The proof is straightforward.

Theorem 3.1 *If S is a commutative Γ-semigroup and $\alpha \in \Gamma$, then the operator semigroup L and the right operator semigroup R of S are isomorphic.*

Proof
Define $f : L \longrightarrow R$ by $f([a,\alpha]) = [\alpha, a]$. Let $[a,\alpha] = [b,\alpha]$. Then $a\alpha s = b\alpha s$ for all $s \in S$. Since S is commutative, $s\alpha a = s\alpha b$ for all $s \in S$. So, $[\alpha, a] = [\alpha, b]$. Thus, f is well-defined. The mapping is injective, since

$$\begin{aligned} [\alpha, a] = [\alpha, b] &\Longrightarrow s\alpha a = s\alpha b \ \ \forall\ s \in S \\ &\Longrightarrow a\alpha s = b\alpha s \\ &\Longrightarrow [a,\alpha] = [b,\alpha] \end{aligned}$$

Again, f is surjective, since for any $[\alpha, a] \in R$, $a \in S$ and $\alpha \in \Gamma$, we have $[\alpha, a] = f[a, \alpha]$.

Also, for all $a, b \in S$ and $\alpha \in \Gamma$,

$$\begin{aligned} f([a,\alpha][b,\alpha]) = f([a\alpha b, \alpha]) = [\alpha, a\alpha b] &= [\alpha, b\alpha a] \\ &= [\alpha, b][\alpha, a] \\ &= [\alpha, a][\alpha, b] \\ &= f([a,\alpha])f([b,\alpha]) \end{aligned}$$

So, f is a homomorphism. Therefore, L and R are isomorphic.

In the following we consider operator semigroups acting on a Γ-semigroup.

Definition 3.1 *Let S be a Γ-semigroup and its left and right operator semigroup are respectively L and R. Then for every $a \in S$ and $\alpha \in \Gamma$, we define $L \times S \longrightarrow S$ and $S \times R \longrightarrow S$ respectively as follow:*

$$[a, \alpha]s := a\alpha s \text{ and } s[\alpha, a] := s\alpha a.$$

The following remark follows from Definition 3.1.

Remark 3.1 *If S is a commutative Γ-semigroup, then $L \times S = S \times R$.*

Example 3.1 *Clearly, Example 2.3 shows that S is commutative and from Example 2.4 it is easy to verify that $L \times S = S \times R$ for every $s \in S$ and $\alpha \in \Gamma$.*

Proposition 3.2 *If A is an ideal of S, then $L \times A$ is an ideal of $L \times S$.*

Proof
Suppose that A is an ideal of S and $[a, \alpha]t \in L \times A$. Then for every $s, t \in S$ and $\beta \in \Gamma$, we have $a\alpha(s\beta t) \in A$. Thus, $[a, \alpha][s, \beta]t = [a\alpha s, \beta]t \in L \times A$. Similarly, we can prove that $[s, \beta][a, \alpha]t = [s\beta a, \alpha]t \in L \times A$. Hence, $L \times A$ is an ideal of $L \times S$.

Theorem 3.2 *There exists an inclusion preserving bijection between the set of all right ideals of S and the set all right ideals of $L \times S$.*

Proof
Suppose that $\mathcal{I}(S)$ and $\mathcal{I}(L \times S)$ sre the sets of all right ideals of S and $L \times S$ respectively. Clearly, the mapping $f : \mathcal{I}(S) \longrightarrow \mathcal{I}(L \times S)$ by $f(I) = L \times I$ is well-defined. Since I is a right ideal of S, $I\Gamma S \subseteq I$. Thus, $I \subseteq L \times I$. On the other hand, since S has a left unity, $L \times I \subseteq (L \times I)\Gamma S \subseteq I$. Thus, $L \times I = I$. Hence, f is bijective. It remains to show that f is an inclusion preserving mapping. Let I and J be two right ideals of S such that $I \subseteq J$. We have to show that $L \times I \subseteq L \times J$. Let $[a, \alpha] \in L$. Then for every $s \in S$ we have $s \in I$ and so $[a, \alpha]s \in L \times I$. Since $I \subseteq J$, we have $s \in J$. Hence, $[a, \alpha]s \in L \times J$. Therefore, $L \times I \subseteq L \times J$. Hence the proof.

Remark 3.2 *Similar characterisations can be proved for right operator semigroup acting on a Γ-semigroup and right Γ-ideal.*

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